Tradeoffs Between Embeddings in Different Models of the Hyperbolic Space

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INTRODUCTION

Motivation:
Hyperbolic embeddings have achieved recent success in capturing hierarchical information (e.g. WordNet). However, such optimizations that come from hyperbolic space are complex, and there is a lack of solid theoretical framework for understanding the tradeoffs that come from employing different hyperbolic models. We wish to elucidate the tradeoffs between such models, both in theory and in practice.

- This project tests with 4 hyperbolic models: **hyperboloid**, **Poincaré disk**, **half-plane**, and **Beltrami-Klein**.
- We experiment with various graphs consisting of various trees, densities, social networks, and cycles.
- We evaluate performance from the following metrics: **running time, loss, distortion**, and **MAP**.

GRAPHS

We synthesized 8 knowledge graphs with NetworkX by choosing parameters for tree radius, tree height, and/or graph density. These graphs fall into 4 broad categories.

Balanced Trees:

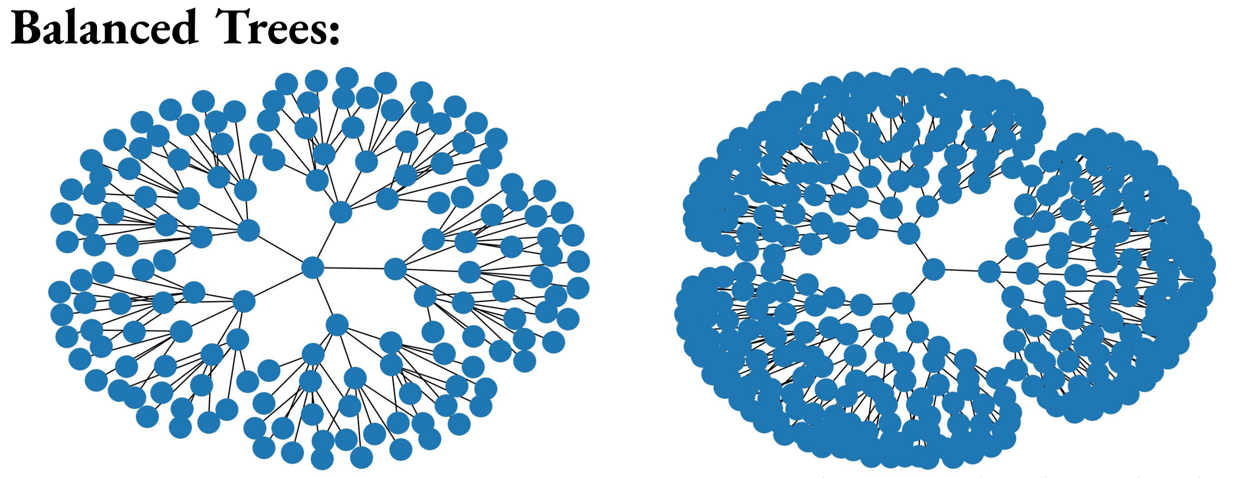


Figure 1b: Tree with radius 5, height 3.

Density Graphs:

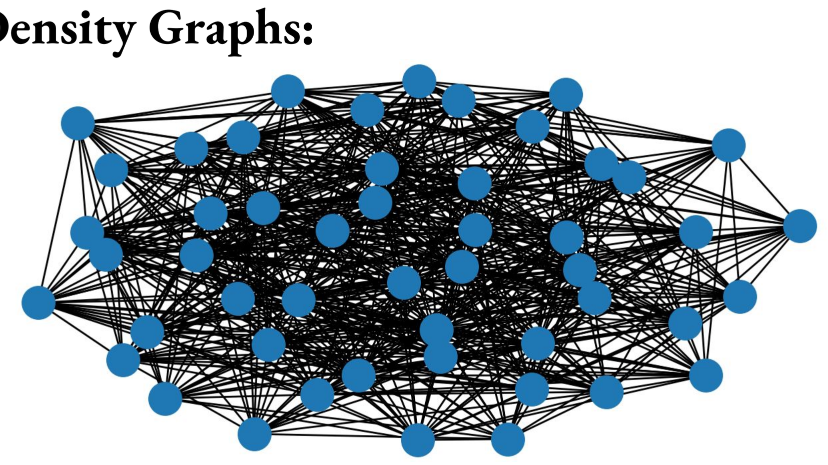


Figure 2: Erdos-Renyl graphs of 400 nodes with densities 0.25, 0.5, 0.75, and 1.

Note: The figure only displays 50 nodes with density 0.5 for clarity.

Social Network Graph:

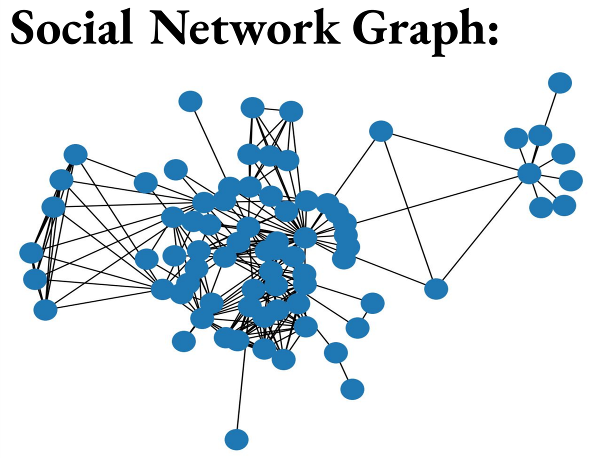


Figure 3: Les Misérables coappearances.

Sierpinski Graph:

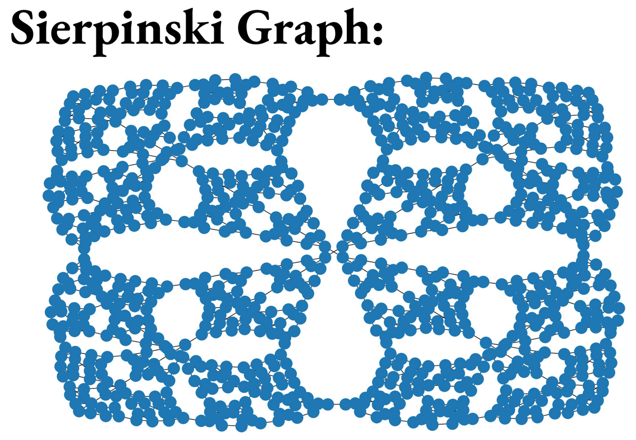


Figure 4: Sierpiński Triangle fractal.

MODELS & EQUATIONS

Models:
where K = Klein, L = Hyperboloid, H = Half-plane, I = Poincaré.

Domains

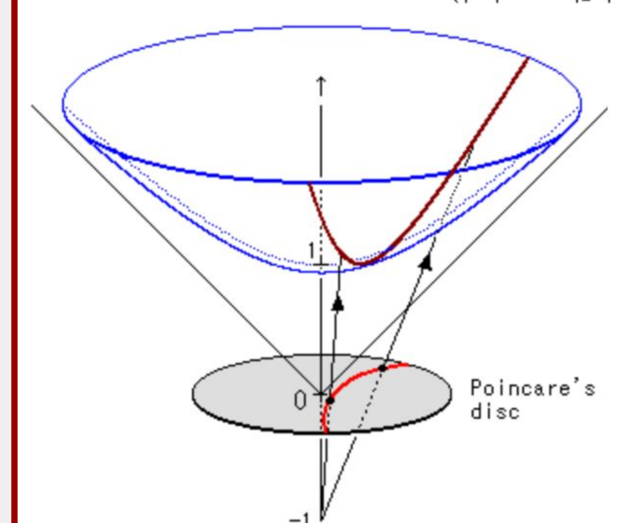
$$K = \{(x_1, \dots, x_n, 1) : x_1^2 + \cdots + x_n^2 < 1\}$$
$$L = \{(x_1, \dots, x_n, x_{n+1}) : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$$
$$H = \{(1, x_2, \dots, x_{n+1}) : x_{n+1} > 0\}$$
$$I = \{(x_1, \dots, x_n, 0) : x_1^2 + \cdots + x_n^2 < 1\}$$

Riemannian Metrics

$$ds_K^2 = \frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)} + \frac{(x_1\,dx_1 + \cdots + x_n\,dx_n)^2}{(1 - x_1^2 - \cdots - x_n^2)^2}$$
$$ds_L^2 = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$$
$$ds_H^2 = \frac{dx_2^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2} \qquad ds_I^2 = 4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - x_1^2 - \cdots - x_n^2)^2}$$

Distance Functions

$$d_K(p,q) = \frac{1}{2}\log\frac{|aq||pb|}{|ap||qb|} \qquad d_L(p,q) = \operatorname{arcosh}(p_1q_1 - p_2q_2 - \dots - p_nq_n)$$
$$d_H(p,q) = \operatorname{arcosh}\left(1 + \frac{(q_1-p_1)^2 + (q_2-p_2)^2 + \dots + (q_n-p_n)^2}{2p_nq_n}\right)$$
$$d_I(p,q) = \operatorname{arcosh}\left(1 + \frac{2|pq|^2|r|^2}{(|r|^2 - |p|^2)(|r|^2 - |q|^2)}\right)$$



From the above equations, we mapped from Euclidean to hyperbolic space. In implementation, we also mapped hyperbolic models between each other.

Figure 5: Demonstrating projection from hyperboloid to Poincaré.

Evaluation Metrics:

Mean Average Precision (MAP)

$$\mathrm{MAP}(f) = \frac{1}{|V|}\sum_{a \in V}\frac{1}{\deg(a)}\sum_{i=1}^{|\mathcal{N}_a|}\frac{|\mathcal{N}_a \cap R_{a,b_i}|}{|R_{a,b_i}|}$$

Distortion

$$D(f) = \frac{1}{\binom{n}{2}}\left(\sum_{u,v \in U : u \neq v}\frac{|d_V(f(u), f(v)) - d_U(u,v)|}{d_U(u,v)}\right)$$

where f: $U \rightarrow V$ with distances d_U, d_V, $a \in V$ with $N_a = \{b_1, b_2, \dots b_{deg(a)}\}$, and $R_{a,bi}$ is the smallest set of nearest points for b_i of $f(a)$.

Loss

$$\mathcal{L}(x) = \sum_{1 \leq i \leq j \leq n}\left|\left(\frac{d_P(x_i, x_j)}{d_G(X_i, X_j)}\right)^2 - 1\right|$$ with graph G and Riemannian manifold P.

RESULTS

Balanced Tree: R=3, H=5						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	10	5	0.12707	0.8811	0.2925	9.57
Poincare	10	5	0.12707	0.8811	0.2925	9.57
Halfplane	**10**	**5**	**0.109614**	**0.8967**	**0.2754**	**9.3912**
Klein	10	0.1	0.310093	0.7703	0.5353	11.8158
Balanced Tree: R=5, H=3						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	10	10	0.004236	0.5888	0.0426	1.5709
Poincare	10	10	0.004236	0.5888	0.0426	1.5709
Halfplane	**10**	**5**	**0.004787**	**0.8641**	**0.046**	**1.6317**
Klein	10	0.1	0.143739	0.8915	0.3439	6.371

Figure 6: Performance on Balanced Trees

Random Graph: 400 Nodes, P=0.25						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	**5**	**5**	**0.003007**	**0.2579**	**0.0259**	**2.2029**
Poincare	**5**	**5**	**0.003007**	**0.2579**	**0.0259**	**2.2029**
Halfplane	10	5	995.505658	0.2632	31.5376	1.2922
Klein	10	0.1	0.000832	0.2475	0.0184	1.4653
Random Graph: 400 Nodes, P=0.50						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	**10**	**5**	**0**	**0.5001**	**0.0001**	**1.0076**
Poincare	5	5	0.001961	0.5049	0.0163	2.0725
Halfplane	10	5	1003.58405	0.5076	31.6572	1.1727
Klein	10	0.1	0.000472	0.494	0.0095	1.4156
Random Graph: 400 Nodes, P=0.75						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	**10**	**5**	**0**	**0.7526**	**0**	**1**
Poincare	**10**	**5**	**0**	**0.7526**	**0**	**1**
Halfplane	10	5	1006.02325	0.7546	31.7221	1.1718
Klein	10	0.1	0.000239	0.7491	0.0051	1.4001
Random Graph: 400 Nodes, P=1.00						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	10	5	0.000341	1	0.0028	1.397
Poincare	10	5	0.000341	1	0.0028	1.397
Halfplane	**10**	**0.1**	**0.000276**	**1**	**0.0013**	**1.4486**
Klein	**10**	**0.1**	**0.000276**	**1**	**0.0013**	**1.4486**

Figure 7: Performance on Erdos-Renyl Random Graphs

Sierpinski: K=4, H=5						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	10	5	0.308715	0.748	0.4733	38.4042
Poincare	10	5	0.308715	0.748	0.4733	38.4042
Halfplane	**10**	**5**	**0.306268**	**0.764**	**0.4684**	**38.453**
Klein	10	0.1	0.583595	0.1828	0.6895	53.2123
Social Graph: Les Miserables						
Model	Dim	Learning Rt	Loss	MAP	Distortion	WC
Hyperboloid	5	5	0.013329	0.8202	0.0869	2.1958
Poincare	5	5	0.013329	0.8202	0.0869	2.1958
Halfplane	**10**	**5**	**0.009316**	**0.9176**	**0.0688**	**2.214**
Klein	10	0.1	0.028229	0.9274	0.1339	2.8731

Figure 8: Performance on Fractal and Social Graph

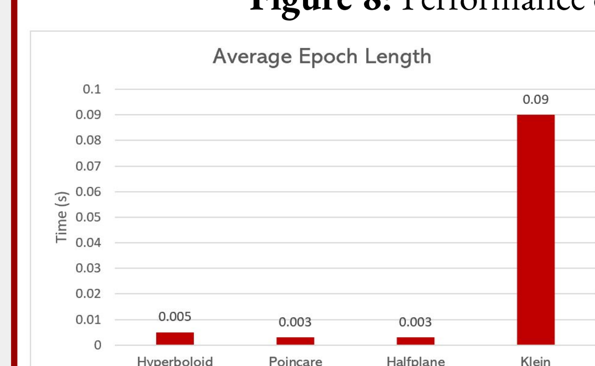


Figure 8: Average Epoch Length

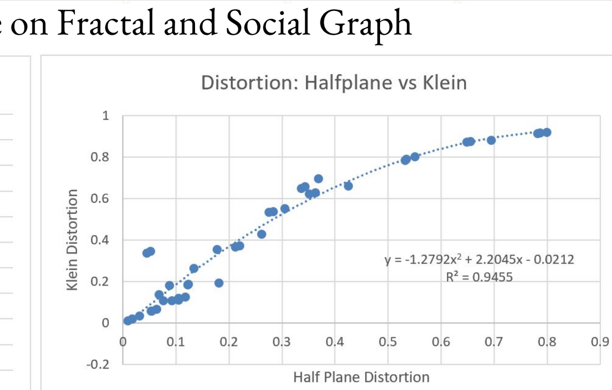


Figure 9: Half-plane vs. Klein Distortion

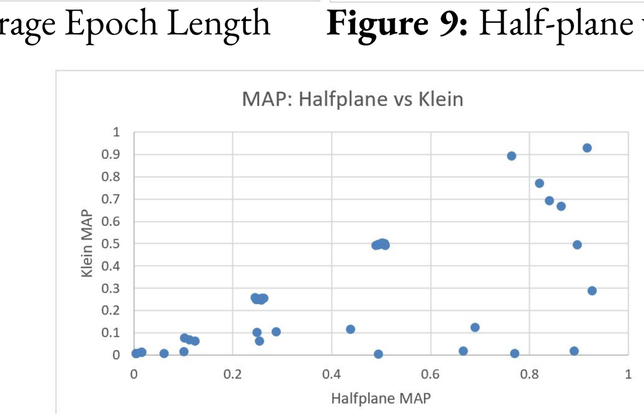


Figure 10: Half-plane vs. Klein MAP

DISCUSSION

- Balanced trees, social graphs, and the last two random trees performed well, although an important factor here is the number of nodes each of these graphs normally has.
- Half-plane and Klein's performances were correlated across runs with the MAP for Half-plane being better implies that Half-plane better preserves the relationship between a given point and its neighbors
- The seeming discrepancy between the MAP and distortion of some of the random graph trials makes sense because the MAP values are pushed towards the density even with a randomly generated embedding.
- Higher dimensions tended to be better since the embedding has more space, but higher learning rates fail faster because approximations in the retractions create magnified errors at the edges.
- Klein takes much longer to train, which is expected because the gradient update is O(n^2) instead of O(n), where n is the dimension, since the more complex metric forces matrix multiplication for the exponential map instead of a rescaling of the gradient. Data shows this: runtimes are off by a factor of 10=100/10

FUTURE WORK

- More fine-grained hyperparameter search to start some predictive effort for learning rate determination
- Analyze computational faults further to attribute more of them to theoretical features of the models
- Test higher dimensional models with much larger datasets to tease out more accurate capabilities for each one
- Compare with Euclidean and Spherical embeddings

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This work was done in collaboration with Frederic Sala and Adva Wolf.

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