A Neural Network Based ElectroMagnetic Solver

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Background and Motivation

Neural Network (NN) based numerical method provides an alternate approach to solving Boundary Value Problems (BVPs) in ElectroMagnetics (EM)[1],[2]. The principal advantages of the Neural Network based numerical method are

- the discrete data points can be unstructured and therefore the issues of meshing are not a factor.
- the solutions are in a differentiable, closed analytic form which avoids the need to interpolate between data points where solutions are obtained using other methods.
- inherently parallel, hence can be efficiently implemented on parallel architectures.

Problem Statement

To formulate, implement and validate a NN based EM field computation for electromagnetic problems with Dirichlet and Mixed boundary condition.

Problem Formulation

BVPs where the unknown field $\psi(x)$ is given by a general second order differential equation are considered in this study.

$$G(x, \psi(x), \nabla\psi(x), \nabla^2\psi(x)) = 0, \quad \forall x \in D$$

The trial solution of the differential equation:

$$\psi_t(x, W, b) = \hat{\psi}(x) + F(x)N(x, W, b)$$

Cost Function is

$$G(x, \psi_t(x, W, b), \nabla\psi_t(x, W, b), \nabla^2\psi_t(x, W, b))^2$$

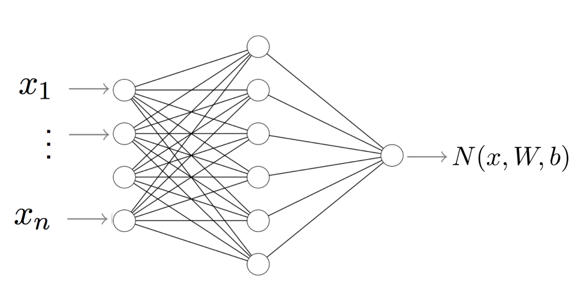


Figure: Neural Network Architecture with H hidden nodes

$$N = W^{[2]}\sigma(h)$$
$$\frac{\partial N}{\partial x_j} = \left(W^{[2]} \circ W_j^{[1]}\right)\sigma^{(1)}(h)$$
$$\frac{\partial^2 N}{\partial x_j^2} = \left(W^{[2]} \circ W_j^{[1]} \circ W_j^{[1]}\right)\sigma^{(2)}(h)$$

Laplace Equation - Dirichlet boundary conditions

$$\nabla^2\psi(x) = 0, \quad \forall x \in D$$

Boundary conditions:

$$\psi(x) = 0, \quad \forall x \in \{(x_1, x_2) \in \{x_1 = 0, x_1 = 1, x_2 = 0\}\}$$
$$\psi(x) = sin(\pi x_1), \quad \forall x \in \{(x_1, x_2) \in \{x_2 = 1\}\}$$

Analytical Solution:

$$\psi_a(x) = \frac{1}{e^{\pi} - e^{-\pi}} sin(\pi x_1)(e^{\pi x_2} - e^{-\pi x_2})$$

NN based method solution:

$$\psi_t(x) = x_2 sin(\pi x_1) + x_1(1 - x_1)x_2(1 - x_2)N(x, W, b)$$

The cost function:

$$-\pi^2 x_2 sin(\pi x_1) + x_2(1 - x_2)\left|x_1(1 - x_1)\frac{\partial^2 N}{\partial x_1^2} + (2 - 4x_1)\frac{\partial N}{\partial x_1} - 2N\right| + x_1(1 - x_1)\left|x_2(1 - x_2)\frac{\partial^2 N}{\partial x_2^2} + (2 - 4x_2)\frac{\partial N}{\partial x_2} - 2N\right|$$

Poisson's Equation - Mixed boundary conditions

$$\nabla^2\psi(x) = (2 - \pi^2 y^2)sin(\pi x), \quad \forall x \in D$$

Boundary conditions:

$$\psi(x) = 0, \quad \forall x \in \{(x_1, x_2) \in \{x_1 = 0, x_1 = 1, x_2 = 0\}\}$$
$$\frac{\partial\psi(x)}{\partial x_2} = 2sin(\pi x_1), \quad \forall x \in \{(x_1, x_2) \in \{x_2 = 1\}\}$$

Analytical solution:

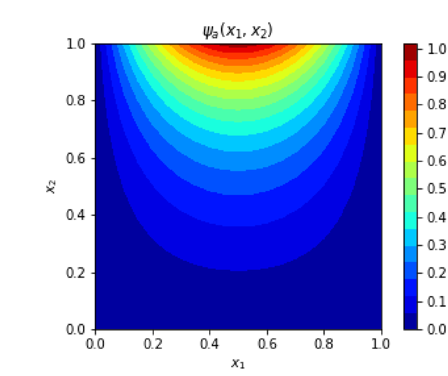
$$\psi_a(x) = x_2^2 sin(\pi x_1)$$

NN based method solution:

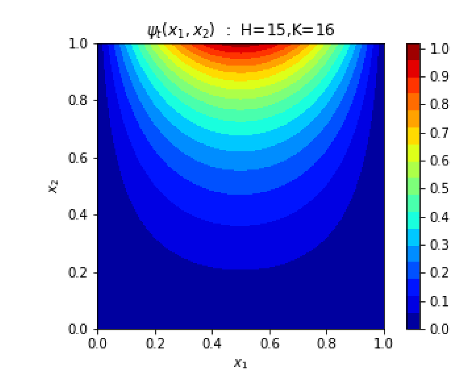
$$\psi_t(x) = 2x_2 sin(\pi x_1) + x_1(1 - x_1)x_2\left[N(x_1, x_2, W, b)\right] - x_1(1 - x_1)x_2\left[N(x_1, 1, W, b)\right] - x_1(1 - x_1)x_2\left[\frac{\partial N(x_1, 1, W, b)}{\partial x_2}\right]$$

Results and Discussion

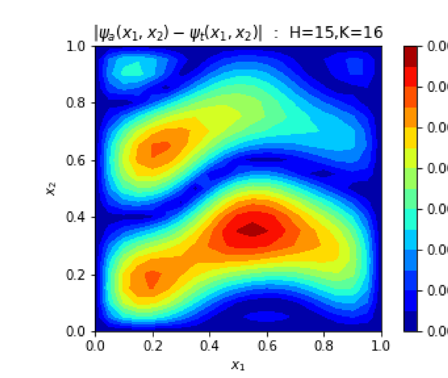
Good agreement between analytical and NN based solution is observed with absolute error of about 0.002 for Laplace equation and 0.09 for Poisson's equation as shown below.



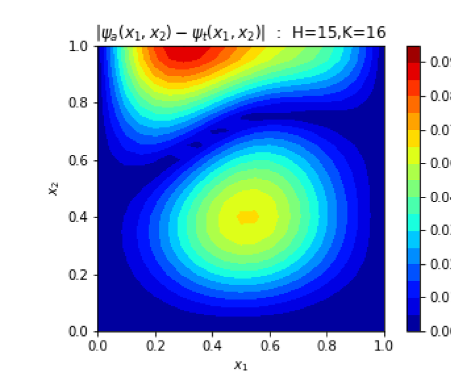
(a) Laplace - Analytical



(b) Laplace - NN based



(c) Laplace-Absolute Error

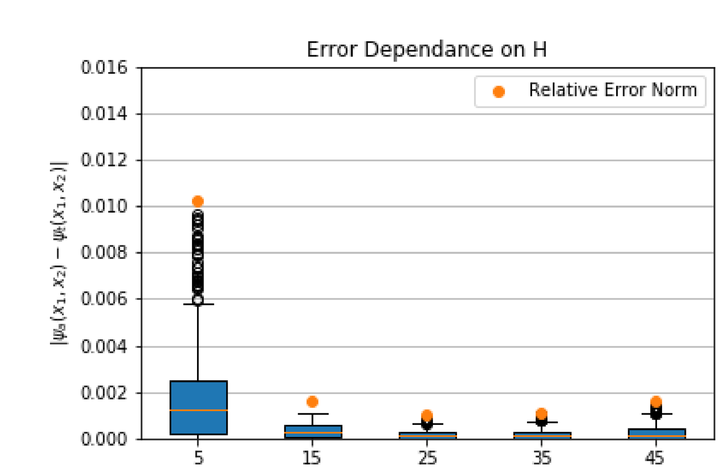


(d) Poisson-Absolute Error

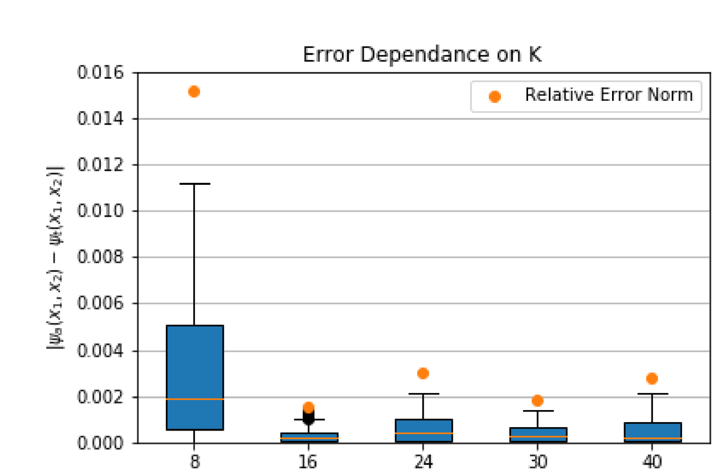
Error metrics in this study:

Absolute Error= $|(\psi_a(x) - \psi_t(x))|$

Relative Error Norm, $E_{norm} = \frac{\sqrt{\Sigma_x(\psi_a(x) - \psi_t(x))^2}}{\sqrt{\Sigma_x(\psi_a(x))^2}}$



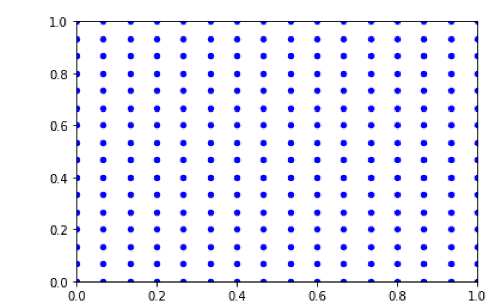
(a) Error vs Number of Hidden Nodes, H



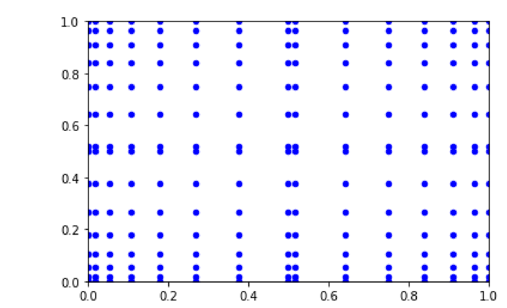
(b) Error vs Training Grid Resolution, K

The relative error norm is lower and the absolute error distribution is tighter with increase in either the number of hidden nodes, H or the training grid resolution, K.

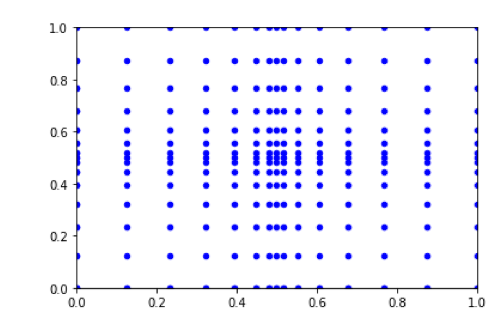
Results and Discussion (Contd...)



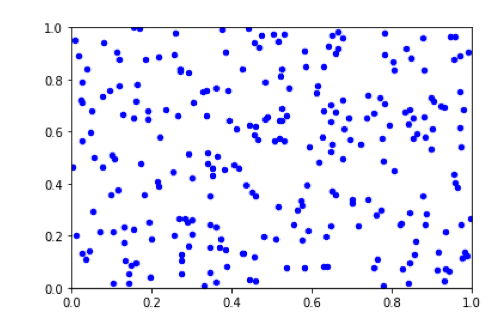
(a) Grid 1: Uniform



(b) Grid 2: Non-Uniform



(c) Grid 3: Non-Uniform



(d) Grid 4: Random

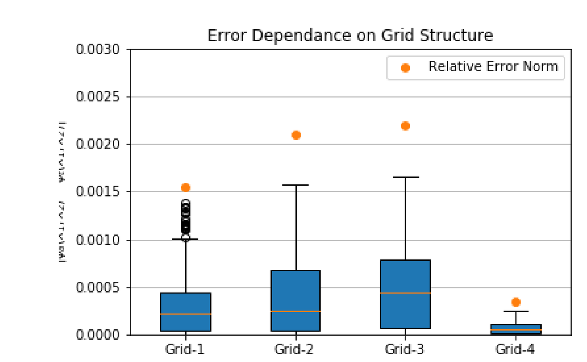


Figure: Error : Structured vs Unstructured Training Data

Summary and Future Work

NN based numerical method has been implemented and validated for two cases. The capability of this method has been showcased by comparing the numerical results with structured and unstructured training data set. Future efforts will be focused on BVPs, where the partial differential equation has complex coefficients, which are commonly found in EM field computation.

References

[1] I. E. Lagaris, A. Likas, and D. I. Fotiadis.
Artificial neural networks for solving ordinary and partial differential equations.
IEEE Transactions on Neural Networks, 9(5):987–1000, Sep 1998.

[2] K. S. McFall and J. R. Mahan.
Artificial neural network method for solution of boundary value problems with exact satisfaction of arbitrary boundary conditions.
IEEE Transactions on Neural Networks, 20(8):1221–1233, Aug 2009.