Review of Probability Theory

Zahra Koochak and Jeremy Irvin

Elements of Probability

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Sample Space Ω

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Event $A \subseteq \Omega$

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Probability Measure $P : \mathcal{F} \to \mathbb{R}$

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If $A_1, A_2, ...$ disjoint set of events ($A_i \cap A_j = \emptyset$ when $i \neq j$), then

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i)$$

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$Val(X) = \{0, 1, ..., 10\}$

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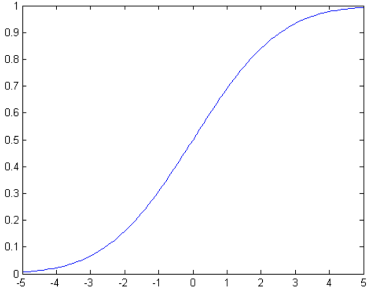

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Variance

$$Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$$

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Variance

$$Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Example Distributions

Distribution	PDF or PMF	Mean	Variance
Bernoulli(p)	$\begin{cases} p, & \text{if } x = 1 \\ 1-p, & \text{if } x = 0. \end{cases}$	p	$p(1-p)$
Binomial(n, p)	$\binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, 1, ..., n$	np	$np(1-p)$
Geometric(p)	$p(1-p)^{k-1}$ for $k = 1, 2, ...$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, ...$	λ	λ
Uniform(a, b)	$\frac{1}{b-a}$ for all $x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Gaussian(μ, σ^2)	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for all $x \in (-\infty, \infty)$	μ	σ^2
Exponential(λ)	$\lambda e^{-\lambda x}$ for all $x \geq 0, \lambda \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Two Random Variables

Bivariate CDF

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

Bivariate PMF

$$p_{XY}(x, y) = P(X = x, Y = y)$$

Marginal PMF

$$p_X(x) = \sum_y p_{XY}(x, y)$$

Bivariate PDF

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

Marginal PDF

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy$$

Bayes' Theorem

- Given the conditional probability of an event $P(x|y)$
- Want to find the "reverse" conditional probability, $P(y|x)$

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

where: $P(x) = \sum_{y' \in value\ y} P(x|y')P(y')$

X and Y are continuous

$$f(y|x) = \frac{f(x|y)f(y)}{f(x)}$$

where: $f(x) = \int_{y' \in value\ y} f(x|y')f(y')dy'$

Example for Bayes Rule

- You randomly choose a treasure chest to open, and then randomly choose a coin from that treasure chest. If the coin you choose is gold, then what is the probability that you choose chest A?

$$a)\frac{1}{3} \qquad b)\frac{2}{3} \qquad c)1 \qquad d)None$$

Independence

Two random variables X and Y are independent if:

- $p_{XY}(x, y) = p_X(x)p_Y(y)$
- $p_{Y|X}(x, y) = P_Y(y)$

For continuous random variables:

$$p_{XY}(x, y) \rightarrow f_{XY}(x, y)$$

Example for independent random variables

- Spin a spinner numbered 1 to 7, and toss a coin. What is the probability of getting an odd. number on the spinner and a tail on the coin?

$$p_{XY}(x, y) = p_X(x)p_Y(y) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$$

Expectation

- X, Y :Two continuous random variables
- g , R2 $\rightarrow$ R : A function of X and Y

$$E(g(x,y)) = \int_{x \in Val(x)} \int_{y \in Val(y)} g(x,y) f_{XY}(x,y) dxdy$$

Example

$g(x,y) = 3x, f_{x,y} = 4xy,\ \ 0 < x < 1,\ \ 0 < y < 1$
$E(g(x,y)) = \int_0^1 \int_0^1 3x \times 4xy\ dxdy$

Covariance of two random variables X and Y

$$Cov[x,y] = E[(x - E[x])(y - (E[y]))]$$
$$= E(XY) - E(X)E(Y)$$

If X and Y are independent, then:

$$E(XY) = E(X)E(Y) \rightarrow Cov[x,y] = 0$$

$$Var[X + Y] = [E(X + Y)]^2 - E((X + Y)^2)$$

$$Var[X + Y] = Var[X] + Var[Y] + 2Cov[X, Y]$$

Multivariant Gaussian (Normal) distribution

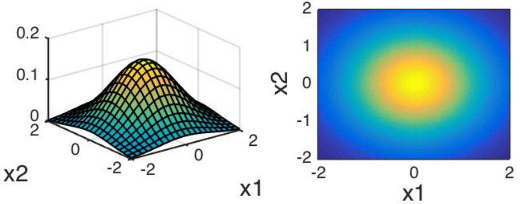
$x \in \mathrm{I\!R^n}$. Model $p(x_1), p(x_2), ...etc$. at the same time. Parameters $:\mu \in \mathrm{I\!R^n}, \Sigma \in \mathrm{I\!R^{n\times n}(covariancematrix)}$

$$p(x;\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{\frac{1}{2}}} exp(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu))$$

Multivariant Gaussian (Normal examples)

$$\Sigma = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\mu = [0 \;\; 0]^T$$



$$\Sigma = \begin{matrix} 0.7 & 0 \\ 0 & 0.7 \end{matrix}$$

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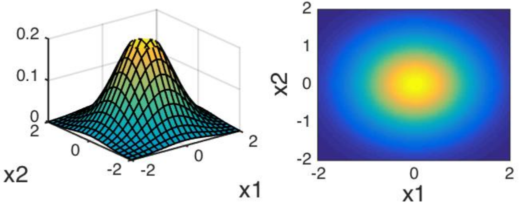

$$\Sigma = \begin{matrix} 1.5 & 0 \\ 0 & 1.5 \end{matrix}$$

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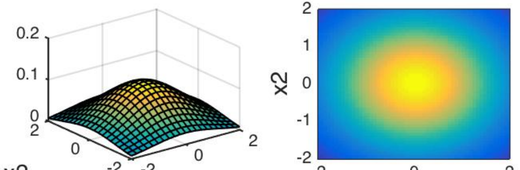

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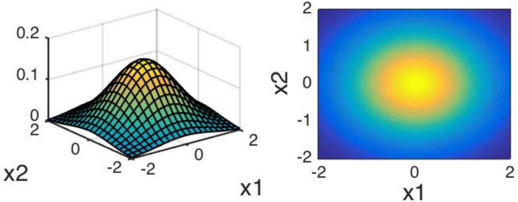

$$\Sigma = \begin{matrix} 0.6 & 0 \\ 0 & 1 \end{matrix}$$

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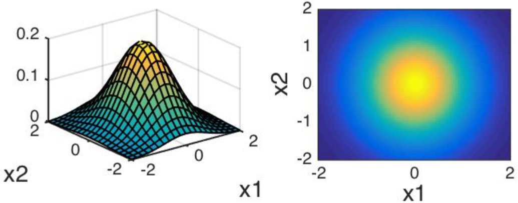
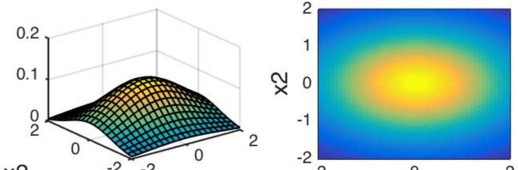


$$\Sigma = \begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}$$

$$\mu = [0 \;\; 0]^T$$



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$$\Sigma = \begin{matrix} 1 & 0.5 \\ 0.5 & 1 \end{matrix}$$

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$$\Sigma = \begin{matrix} 1 & 0.8 \\ 0.8 & 1 \end{matrix}$$

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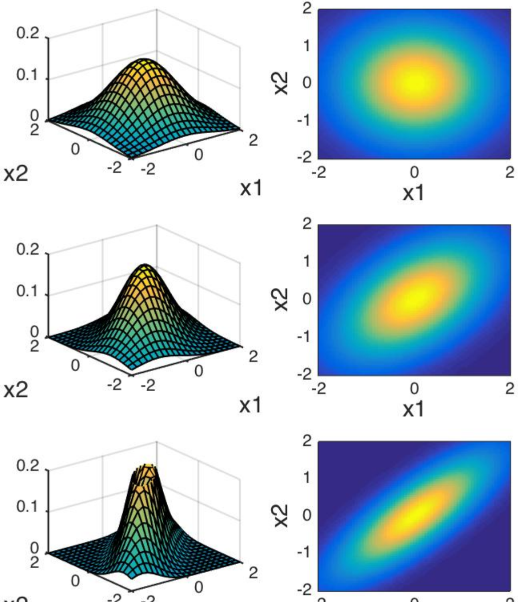

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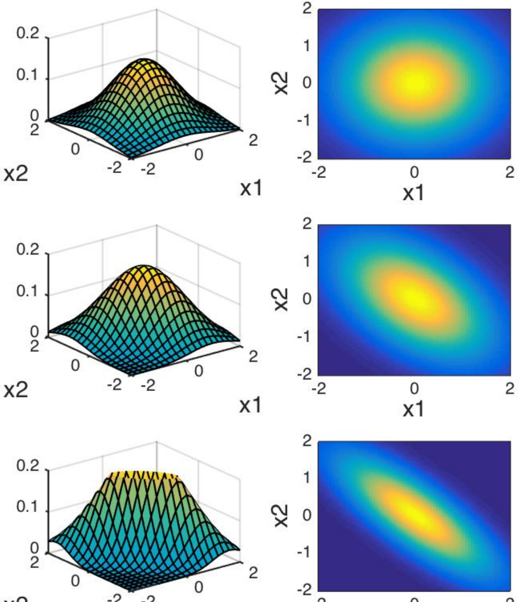

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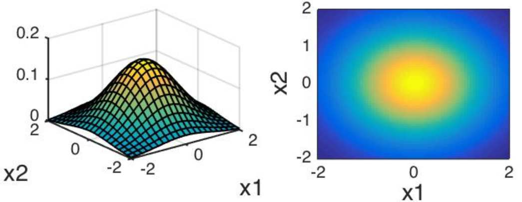

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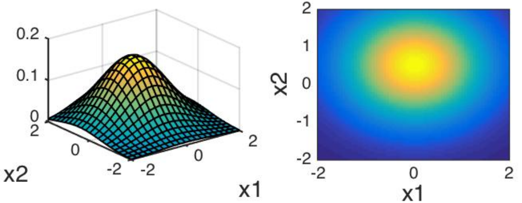

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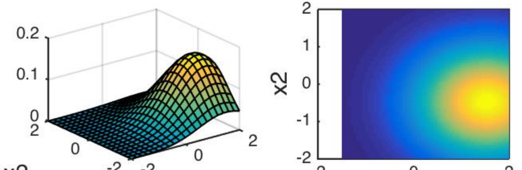

Conditional Probability and Expectation

Remember:

Let B be any event such that $P(B) \neq 0$.

$P(A|B) := \frac{P(A \cap B)}{P(B)}$

Conditional Probability and Expectation

X,Y are RVs with the same probability space,

we have

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$\mathbb{E}(X|Y = y) = \sum_x x \frac{P(X = x, Y = y)}{P(Y = y)}$$

Conditional Probability and Expectation

$$\mathbb{E}[X \mid Y]$$

Conditional Probability and Expectation

$\mathbb{E}[X|Y]$

It is actually a random variable

$\mathbb{E}[X|Y](y) = \mathbb{E}[X|Y = y]$ is a function of Y

Law of Total Expectation

Let X, Y be RVs with the same probability space, then

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$

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A brief proof of X,Y being discrete and finite

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A brief proof of X,Y being discrete and finite

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[X|Y]] &= \mathbb{E}[\sum_x xP(X = x|Y)] \\
&= \sum_y (\sum_x xP(X = x|Y = y))P(Y = y) \\
&= \sum_y \sum_x xP(X = x, Y = y) \\
&= \sum_x x(\sum_y P(X = x, Y = y)) \\
&= \sum_x xP(X = x) \\
&= \mathbb{E}[X]
\end{aligned}$$

More Conditioned Bayes Rule

$$P(a|b,c) = \frac{P(b|a,c)P(a|c)}{P(b|c)}$$

It is actually the same as the Bayes Rule:

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

with a random variable c that all the probabilities are conditioned on.

More Conditioned Bayes Rule

A proof:

$$\begin{aligned}
\frac{P(b|a,c)P(a|c)}{P(b|c)} &= \frac{P(b,a,c)P(a|c)}{P(b|c)P(a,c)} \\
&= \frac{P(b,a,c)P(a,c)}{P(b|c)P(a,c)P(c)} \\
&= \frac{P(b,a,c)}{P(b|c)P(c)} \\
&= \frac{P(b,a,c)}{P(b,c)} \\
&= P(a|b,c)
\end{aligned}$$