Mesh Sampling for Finite Element Method

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Introduction

Finite Element Method (FEM) plays an important role in solving partial differential equations. Solid mechanics and structural dynamics is one application of FEM, and it has the governing equation in semi-discrete form in equation 1.

$$M\ddot{u} + f(u) = f^{\text{ext}} \tag{1}$$

The most time consuming part of solving equation 1 is the operator $f(u)$, especially in nonlinear case. We assemble $f^e(u)$ from discrete elements to obtain $f(u)$ by equation 2.

$$f(u) = \sum_{e \in \mathcal{V}} f^e(u) \tag{2}$$

Method

To reduce the assembly cost, we sample a subset of elements $\mathcal{V}'$ form $\mathcal{V}$ and attribute them non-negative weights $\alpha^e > 0 \; \forall e \in \mathcal{V}'$ such that

$$f(u) \approx \sum_{e \in \mathcal{V}'} \alpha^e f^e(u) \tag{3}$$

However, $f^e(u)$ is a sparse vector since it only contains the local information. Thus, the assembly of sampled elements only produces a sparse internal force vector as well. Alternatively, we can replace it by the weighting matrix W^e to broadcast local information, as shown in equation 4.

$$\hat{f}(u) = \sum_{e \in \mathcal{V}'} W^e \hat{f}^e(u) \tag{4}$$

Data

Training set contains $\{f^{(i)}, (\hat{f}^{1,(i)}, \hat{f}^{2,(i)}, \ldots, \hat{f}^{|\mathcal{V}|,(i)})\}$, for $i = 1, 2, \ldots, n$ and validation set contains $\{f^{(j)}, (\hat{f}^{1,(j)}, \hat{f}^{2,(j)}, \ldots, \hat{f}^{|\mathcal{V}|,(j)})\}$, for $j = 1, 2, \ldots, m$.

Algorithm

Solve optimization problem in equation 5

$$\mathcal{V}'^*, W^* = \arg\min_{\mathcal{V}', W} \sum_{i=1}^{n} \text{loss}(f^{(i)}, \sum_{e \in \mathcal{V}'} W^e \hat{f}^{e,(i)}) \tag{5}$$

where $W = \{W^e\} \; \forall e \in \mathcal{V}'$, and loss function is the 2-norm square of the difference, i.e., $\text{loss}(x, y) = ||x - y||_2^2$.

Suppose the optimal subset of elements $\mathcal{V}'^*$ is known, solve the optimization problem in equation 6.

$$\begin{aligned} W^* &= \arg\min_{W} J_{\text{train}}(\mathcal{V}'^*) \\ &= \arg\min_{W} \sum_{i=1}^{n} \text{loss}(f^{(i)}, \sum_{e \in \mathcal{V}'^*} W^e \hat{f}^{e,(i)}) + \lambda \sum_{e \in \mathcal{V}'^*} ||W^e||_F^2 \end{aligned} \tag{6}$$

The quality of sampled elements is evaluated by equation 7, which computes the relative global error in validation set.

$$J_{\text{val}}(\mathcal{V}'^*, W^*) = \sqrt{\frac{\sum_{j=1}^{m} \text{loss}(f^{(j)}, \sum_{e \in \mathcal{V}'^*} W^{e*} \hat{f}^{e,(j)})}{\sum_{j=1}^{m} ||f^{(j)}||_2^2}} \tag{7}$$

Algorithm 1: Mesh sampling (greedy)

initialize $\mathcal{V}' \leftarrow \varnothing$;
while not converged **do**
  (a) Random sample $\mathcal{V}_s \subset \mathcal{V}$;
  (b) For $i \in \mathcal{V}_s$ if $i \notin \mathcal{V}'$, let $\mathcal{V}'_i \leftarrow \mathcal{V}' \cup \{i\}$, compute $W_i^* \leftarrow \arg\min J_{\text{train}}$ and evaluate $J_i = J_{\text{val}}(\mathcal{V}'_i, W_i^*)$;
  (c) Set $\mathcal{V}'$ to be the best subset of elements which minimizes J_{val} on step (b), $\mathcal{V}' \leftarrow \arg\min J_i$;
end

Application

We are interested in a dynamic impact problem of a three-dimensional circular bar on a rigid frictionless wall. Material response of the bar is characterized by a model of J_2 flow theory using a logarithmic free energy function and the von Mises yield criterion. Because of the symmetry, only one quarter of the bar is modeled. 1728 20-noded hexahedral elements are used with displacement-only DOFs. The initial velocity of the bar is 227 m/s. The temporal discretization is performed using the explicit central difference method with variable time-step based on the estimation of the critical stability time-step. The simulation runs until $T = 5 \times 10^{-5}$ s.

Material properties		Geometry	
Young's modulus E	117 GPa	Length L	32.4 mm
Poisson's ratio ν	0.4	Radius of circular cross-section R	3.2 mm
Yield stress σ_y	0.4 GPa		
Isotropic hardening modulus H	0.1 GPa		
Density ρ	8930 kg/m^3		

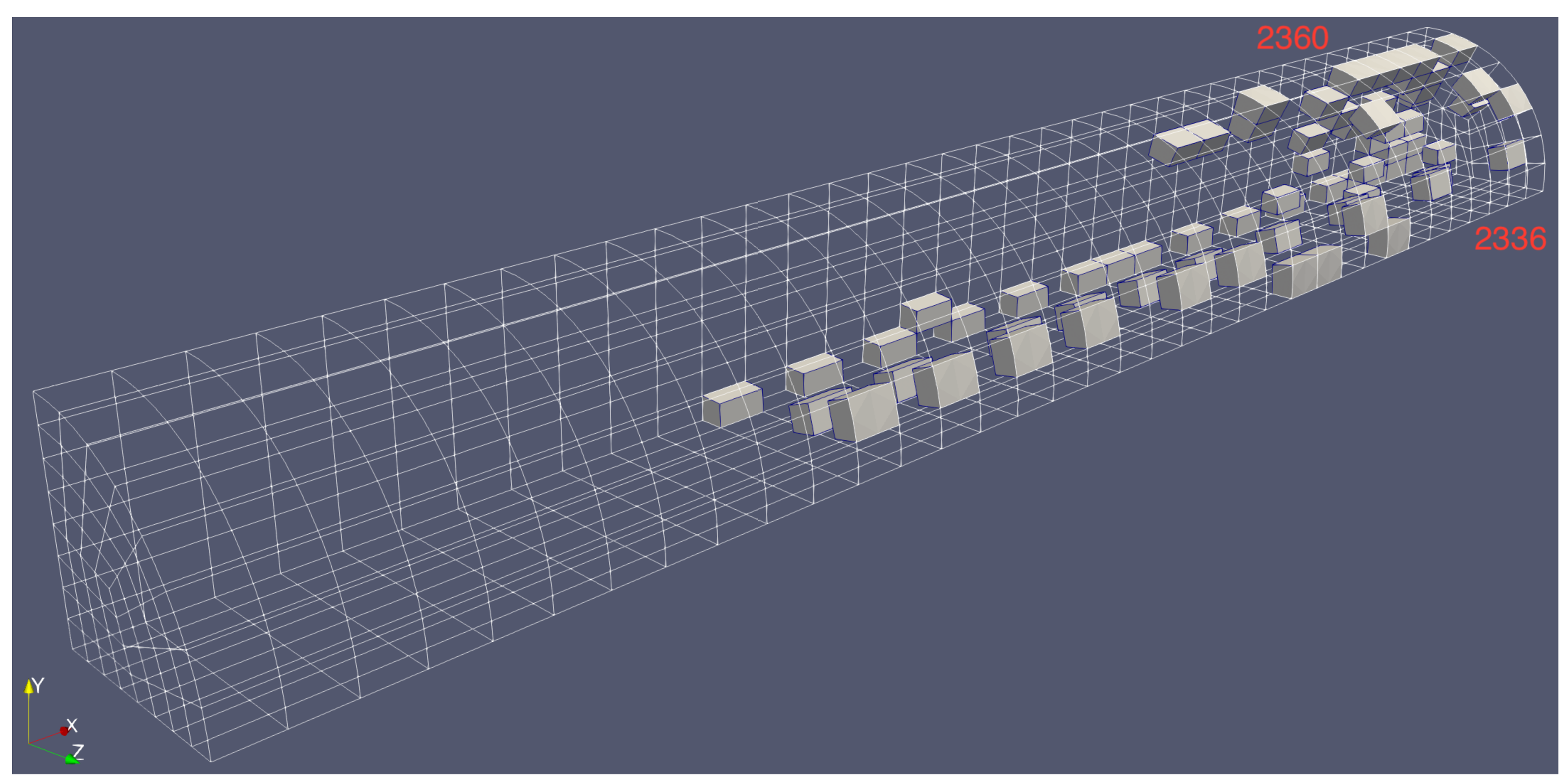

Fig. 1: Circular bar: mesh (white lines), sampled elements (white solid elements), and probe locations

Result

	Efficiency		Accuracy	
	Model I	Model II		Global relative errors (%)
CPU times (s)	1.8×10^3	5.64×10^2	x-displacement	0.46
Speed-up factors	1.0	3.2	y-displacement	0.92
$\lvert\mathcal{V}\rvert$	1728	70	z-displacement	0.93
			x-velocity	4.0
			y-velocity	9.6
			z-velocity	9.6

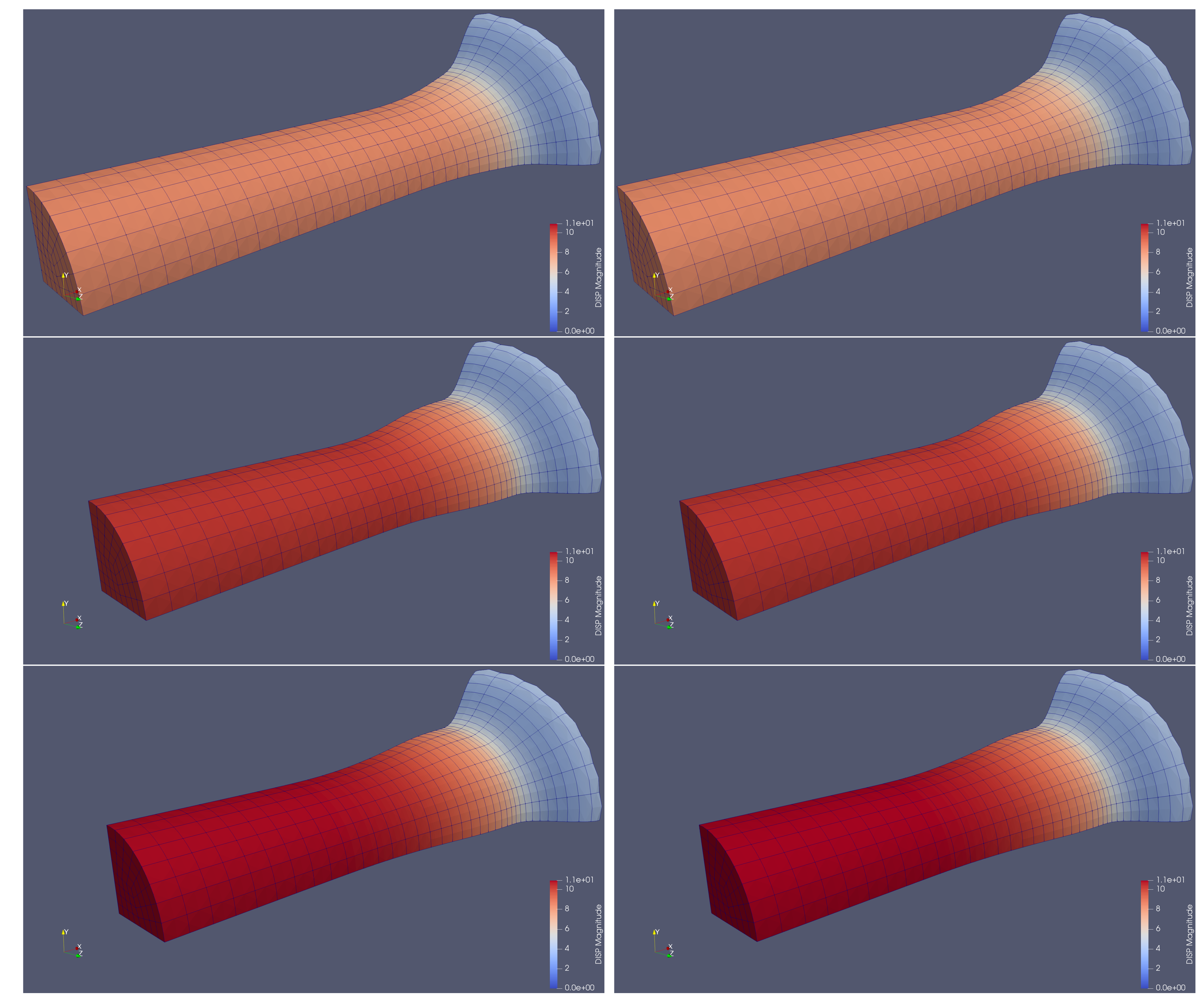

Fig. 2: Impact of circular bar: sequence of deformed displacement field at $t = 44$ μs, $t = 63$ μs, and $t = 80$ μs of Model I (left column) and Model II (right column)

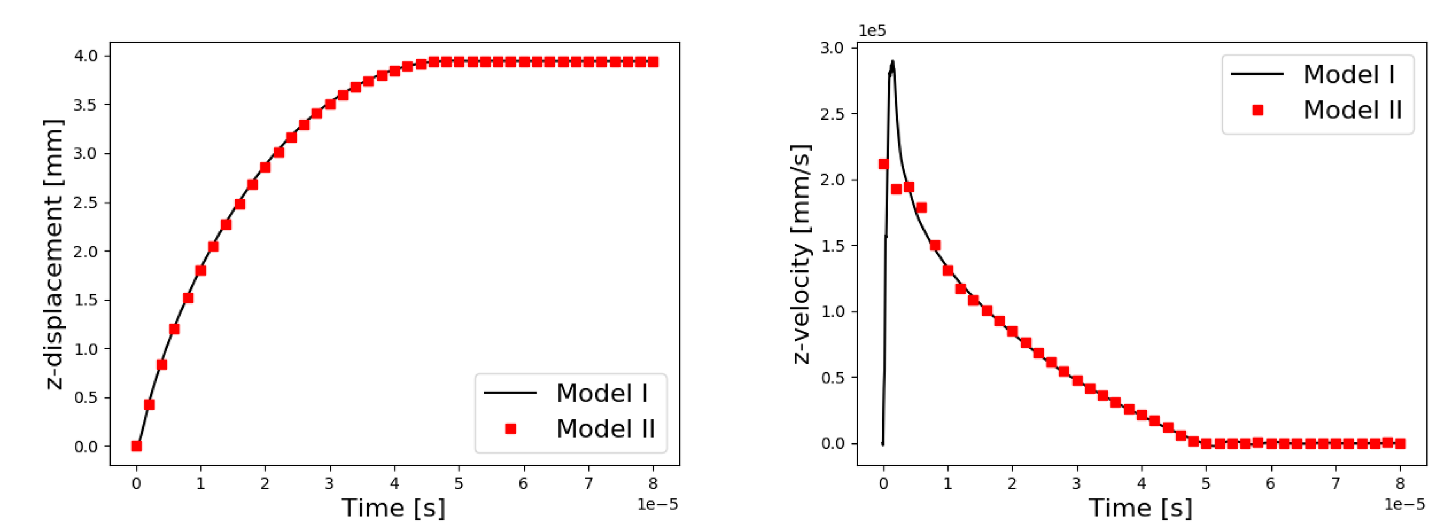

Fig. 3: Impact of circular bar: displacements and velocities in z-direction at node 2336